18. If $u = \sqrt{x^2 + y^2 + z^2}$, show that

(i) $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 1$ (ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{u}$.

19. If $u = e^{x-at} \cos (x - at)$, show that $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$.

20. If $u = e^x(x \cos y - y \sin y)$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

21. If $u = \log (x^3 + y^3 - x^2y - xy^2)$, show that $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -4(x + y)^{-2}$.

[Hint. $u = \log \{x^2(x - y) - y^2(x - y)\} = \log (x - y)(x^2 - y^2) = \log (x - y)^2(x + y) = 2 \log (x - y) +$ $\log (x + y)$]

22. Show that the function $f(x, y) = \begin{cases} \frac{x^2 + y^2}{|x| + |y|} & ; (x, y) \neq 0 \\ 0 & ; (x, y) = 0 \end{cases}$ is continuous at (0, 0) but its partial derivatives do not exist at (0, 0).

23. For the function $f(x, y) = \begin{cases} \frac{xy(2x^2 - 3y^2)}{x^2 + y^2} & ; (x, y) \neq (0, 0) \\ 0 & ; (x, y) = (0, 0) \end{cases}$

find $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ and prove that f_{xy}, f_{yx} are discontinuous at (0, 0).

Answers

1. (i) $y^x \log y,\ xy^{x-1}$; (ii) $\frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2}$ (iii) $2x \sin \frac{y}{x} - y \cos \frac{y}{x}, x \cos \frac{y}{x}$

(iv) $\frac{-x}{x^2 + y^2} + \frac{1}{y} \tan^{-1} \frac{y}{x}, \frac{x^2}{y(x^2 + y^2)} - \frac{x}{y^2} \tan^{-1} \frac{y}{x}$ 13. $n = -3, 2$.

23. $f_{xy}(0, 0) = 0, f_{yx}(0, 0) = -3$.

(P.T.U., May 2005)

3.5. HOMOGENEOUS FUNCTIONS

A function $f(x, y)$ is said to be homogeneous of degree (or order) n in the variables x and y if it can be expressed in the form $x^n \phi\left(\frac{y}{x}\right)$ or $y^n \phi\left(\frac{x}{y}\right)$.

An alternative test for a function $f(x, y)$ to be homogeneous of degree (or order) n is that

$$f(tx, ty) = t^n f(x, y).$$

For example, if $f(x, y) = \frac{x + y}{\sqrt{x} + \sqrt{y}}$, then

(i) $$f(x, y) = \frac{x\left(1 + \frac{y}{x}\right)}{\sqrt{x}\left(1 + \sqrt{\frac{y}{x}}\right)} = x^{1/2} \phi\left(\frac{y}{x}\right)$$

$\Rightarrow$ $f(x, y)$ is a homogeneous function of degree $\frac{1}{2}$ in x and y.

$$(ii)\ f(x, y) = \frac{y\left(\frac{x}{y}+1\right)}{\sqrt{y}\left(\sqrt{\frac{x}{y}}+1\right)} = y^{1/2}\,\phi\left(\frac{x}{y}\right)$$

$\Rightarrow$ $f(x, y)$ is a homogeneous function of degree $\frac{1}{2}$ in x and y.

$$(iii)\ f(tx, ty) = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = \frac{t(x+y)}{\sqrt{t}\,(\sqrt{x}+\sqrt{y})} = t^{1/2} f(x, y)$$

$\Rightarrow$ $f(x, y)$ is a homogeneous function of degree $\frac{1}{2}$ in x and y.

Similarly, a function $f(x, y, z)$ is said to be homogeneous of degree (or order) n in the variables x, y, z if

$$f(x, y, z) = x^n\,\phi\left(\frac{y}{x}, \frac{z}{x}\right) \quad \text{or} \quad y^n\,\phi\left(\frac{x}{y}, \frac{z}{y}\right) \quad \text{or} \quad z^n\,\phi\left(\frac{x}{z}, \frac{y}{z}\right).$$

Alternative test is $f(tx, ty, tz) = t^n f(x, y, z)$.

3.6. EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

(P.T.U. May 2002, May 2003, Dec. 2004, May 2005, May 2006)

If u is a homogeneous function of degree n in x and y, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu.$

Proof. Since u is a homogeneous function of degree n in x and y, it can be expressed as

$$u = x^n f\left(\frac{y}{x}\right)$$

$$\therefore \quad \frac{\partial u}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right).\left(-\frac{y}{x^2}\right)$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} = nx^n f\left(\frac{y}{x}\right) - x^{n-1} y f'\left(\frac{y}{x}\right) \qquad \ldots(1)$$

$$\text{Also} \quad \frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right).\frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad y\frac{\partial u}{\partial y} = x^{n-1} y f'\left(\frac{y}{x}\right) \qquad \ldots(2)$$

Adding (1) and (2), we get $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nx^n f\left(\dfrac{y}{x}\right) = nu.$

Note. Euler's theorem can be extended to a homogeneous function of any number of variables. Thus, if u is a homogeneous function of degree n in x, y and z, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = nu.$

3.7. IF u IS A HOMOGENEOUS FUNCTION OF DEGREE n IN x AND y, THEN

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

(*Karnataka, 1990 ; Mangalore, 1997*)

Proof. Since u is a homogeneous function of degree n in x and y

∴ By Euler's Theorem, we have $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$ ...(1)

Differentiating (1) partially w.r.t. x, we have $1 . \frac{\partial u}{\partial x} + x \frac{\partial^2 u}{\partial x^2} + y . \frac{\partial^2 u}{\partial x \partial y} = n \frac{\partial u}{\partial x}$ ...(2)

Differentiating (1) partially w.r.t. y, we have $x \frac{\partial^2 u}{\partial y \partial x} + 1 . \frac{\partial u}{\partial y} + y . \frac{\partial^2 u}{\partial y^2} = n . \frac{\partial u}{\partial y}$

But $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

∴ $$x \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y \frac{\partial^2 u}{\partial y^2} = n \frac{\partial u}{\partial y} \quad ...(3)$$

Multiplying (2) by x, (3) by y and adding

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = n \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right)$$

or $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + nu = n . nu \quad \text{[Using (1)]}$$

⇒ $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n^2 u - nu = n(n-1)u.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Verify Euler's theorem for the functions :*

(*i*) $u = (x^{1/2} + y^{1/2})(x^n + y^n)$

(*ii*) $u = \sin^{-1} \frac{x}{y} + \tan^{-1} \frac{y}{x}$ (P.T.U. May 2002 ; Mysore 1994 ; J.N.T.U. 1990)

(*iii*) $f(x, y, z) = 3x^2yz + 5xy^2z + 4z^4$ (P.T.U. Dec., 2005)

Sol. (*i*) $u = (x^{1/2} + y^{1/2})(x^n + y^n)$ ...(1)

$$= x^{1/2}\left(1 + \frac{y^{1/2}}{x^{1/2}}\right) x^n \left(1 + \frac{y^n}{x^n}\right) = x^{n+1/2}\left[1 + \left(\frac{y}{x}\right)^{1/2}\right]\left[1 + \left(\frac{y}{x}\right)^n\right] = x^{n+1/2} f\left(\frac{y}{x}\right)$$

[OR $f(tx, ty) = t^{n+1/2} f(x, y)$]

⇒ u is a homogeneous function of degree $\left(n + \frac{1}{2}\right)$ in x and y

∴ By Euler's theorem, we should have $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \left(n + \frac{1}{2}\right) u$ ...(2)

From (1), $$\frac{\partial u}{\partial x} = \frac{1}{2} x^{-1/2} (x^n + y^n) + nx^{n-1}(x^{1/2} + y^{1/2})$$

$$x\frac{\partial u}{\partial x} = \frac{1}{2}x^{1/2}(x^n + y^n) + nx^n(x^{1/2} + y^{1/2})$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}y^{-1/2}(x^n + y^n) + ny^{n-1}(x^{1/2} + y^{1/2})$$

$$y\frac{\partial u}{\partial y} = \frac{1}{2}y^{1/2}(x^n + y^n) + ny^n(x^{1/2} + y^{1/2})$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}(x^{1/2} + y^{1/2})(x^n + y^n) + n(x^n + y^n)(x^{1/2} + y^{1/2})$$

$= \frac{1}{2}u + nu = (n + \frac{1}{2})\,u$ which is the same as (2). Hence the verification.

(*ii*) $$u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x} \qquad ...(1)$$

$$= \text{cosec}^{-1}\frac{y}{x} + \tan^{-1}\frac{y}{x} = x^0 f\left(\frac{y}{x}\right)$$

[OR $f(tx, ty) = f(x, y) = t^{\circ} f(x, y)$]

$\Rightarrow$ u is a homogeneous function of degree 0 in x and y.

$\therefore$ By Euler's theorem, we should have $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0 \times u = 0$ ...(2)

From (1), $$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\cdot\frac{1}{y} + \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}$$

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(-\frac{x}{y^2}\right) + \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} = -\frac{x}{y\sqrt{y^2 - x^2}} + \frac{x}{x^2 + y^2}$$

$$y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2}$$

$\therefore \quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$ which is the same as (2). Hence the verification.

(*iii*) $f(x, y, z) = 3x^2yz + 5xy^2z + 4z^4$ ...(1)

$f(x, y, z)$ is a homogeneous function of x, y, z of degree 4.

By Euler's theorem $x\,.\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} + z\dfrac{\partial f}{\partial z} = nf = 4f$

Differentiable (1) partially w.r.t. x, y, z successively, we get

$$\frac{\partial f}{\partial x} = 6xyz + 5y^2z$$

$$\frac{\partial f}{\partial y} = 3x^2z + 10xyz$$

$$\frac{\partial f}{\partial z} = 3x^2y + 5xy^2 + 16z^3$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 6x^2yz + 5xy^2z + 3x^2yz + 10xy^2z + 3x^2yz + 5xy^2z + 16z^4$$

$$= 12x^2yz + 20xy^2z + 16z^4 = 4(3x^2yz + 5xy^2z + 4z^4) = 4f(x, y, z).$$

Example 2. *(i) If* $u = f\left(\frac{y}{x}\right)$ *show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$ (P.T.U. Dec. 2000)

(ii) If $u = x f\left(\frac{y}{x}\right)$, *show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u.$

Sol. (*i*) $u = f\left(\frac{y}{x}\right)$

$= x^\circ f\left(\frac{y}{x}\right)$, which is a homogeneous function of degree 0

∴ By Euler's theorem $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\,u = 0.u = 0$

Hence $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$

(*ii*) $u = x f\left(\frac{y}{x}\right)$; it is a homogeneous function of degree 1

∴ By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1\,.\,u = u.$$

Example 3. *If* $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$; *find the value of* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

(Mysore 1994; J.N.T.U. 1990)

Sol. Let $u = f(x, y)$

∴ $f(x, y) = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$

$$f(tx, ty) = \sin^{-1}\frac{tx}{ty} + \tan^{-1}\frac{ty}{tx} = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x} = f(x, y)$$

∴ $f(tx, ty) = t^\circ f(x, y)$

∴ $f(x, y)$ is a homogeneous function of x and y of degree 0

∴ By Euler's theorem $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0\,.\,f = 0$

or $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

[or the homogenity of u can be shown like this

$$u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x} = \text{cosec}^{-1}\frac{y}{x} + \tan^{-1}\frac{y}{x}$$

$$= x^0 \,[\text{cosec}^{-1} y/x + \tan^{-1} y/x]$$

which is homogeneous function of degree 0.]

Example 4. (*a*) *If* $u = \tan^{-1} \dfrac{x^3 + y^3}{x - y}$, *prove that* $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = \sin 2u$.

(A.M.I.E. 1990 ; Kerala, 1990)

(*b*) *If* $f(x, y) = \dfrac{1}{x^2} + \dfrac{1}{xy} + \dfrac{\log x - \log y}{x^2 + y^2}$, *show that* $x \dfrac{\partial f}{\partial x} + y \dfrac{\partial f}{\partial y} = -2f$.

(P.T.U., May 2004)

Sol. (*a*) Here u is not a homogeneous function

$$\left(\because \quad u = \tan^{-1} x^2 \frac{1 + \left(\frac{y}{x}\right)^3}{1 - y/x} \text{ cannot be expressed as } x^n f\left(\frac{y}{x}\right) \right)$$

But $$\tan u = \frac{x^3 + y^3}{x - y} = \frac{x^3\left[1 + \left(\frac{y}{x}\right)^3\right]}{x\left[1 - \frac{y}{x}\right]} = x^2 f\left(\frac{y}{x}\right)$$

is a homogeneous function of degree 2 in x and y.

$\therefore$ By Euler's theorem, we have

$$x \frac{\partial}{\partial x} (\tan u) + y \frac{\partial}{\partial y} (\tan u) = 2 \tan u \quad \text{or} \quad x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2 \tan u$$

or $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \sin u}{\cos u} \,.\, \cos^2 u = 2 \sin u \cos u = \sin 2u.$$

(*b*) $$f(x, y) = \frac{1}{x^2} \left[1 + \frac{x}{y} + \frac{\log x/y}{1 + \frac{y^2}{x^2}} \right]$$

$$= \frac{1}{x^2} \left[1 + \frac{1}{y/x} - \frac{\log y/x}{1 + \frac{y^2}{x^2}} \right] = x^{-2} \phi \left(\frac{y}{x}\right)$$

$\therefore$ $f(x, y)$ is a homogeneous function of degree -2

$\therefore$ By Euler's theorem

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf$$

Here $n = -2$

$\therefore$ $$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = -2f.$$

Example 5. *(i) If* $f(x, y) = \sqrt{x^2 - y^2} \sin^{-1} \frac{y}{x}$, *prove that* $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = f(x, y)$.

(ii) If $f(x, y) = \sqrt{y^2 - x^2} \sin^{-1} \frac{x}{y} + \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}$, *show that* $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} - f(x, y) = 0$.

Sol. (*i*)
$$f(x, y) = \sqrt{x^2 - y^2} \sin^{-1} \frac{y}{x}$$
$$= x \sqrt{1 - \left(\frac{y}{x}\right)^2} \sin^{-1} \frac{y}{x}$$
$$= x \left\{ \sqrt{1 - \left(\frac{y}{x}\right)^2} \sin^{-1} \frac{y}{x} \right\} = x \,\varphi\left(\frac{y}{x}\right)$$

which is a homogeneous function of degree 1.

$\therefore$ By Euler's theorem $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f(x, y)$

i.e.,
$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 1 \,.\, f(x, y)$$

Hence
$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = f(x, y)$$

(*ii*)
$$f(x, y) = \sqrt{y^2 - x^2} \sin^{-1} \frac{x}{y} + \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}$$
$$= x \sqrt{\left(\frac{y}{x}\right)^2 - 1} \operatorname{cosec}^{-1} \frac{y}{x} + x \,.\, \frac{1 - \left(\frac{y}{x}\right)^2}{\sqrt{1 + \left(\frac{y}{x}\right)^2}}$$
$$= x \left\{ \sqrt{\left(\frac{y}{x}\right)^2 - 1} \operatorname{cosec}^{-1} \frac{y}{x} + \frac{1 - \left(\frac{y}{x}\right)^2}{\sqrt{1 + \left(\frac{y}{x}\right)^2}} \right\}$$

$= x \,\varphi\left(\frac{y}{x}\right)$ which is homogeneous function of degree 1.

$\therefore$ By Euler's theorem $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 1 \,.\, f(x, y)$

or
$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} - f(x, y) = 0$$

Aliter:
$$f(tx, ty) = \sqrt{t^2 y^2 - t^2 x^2} \sin^{-1} \frac{tx}{ty} + \frac{t^2 x^2 - t^2 y^2}{\sqrt{t^2 x^2 + t^2 y^2}}$$

$$= t\sqrt{y^2 - x^2}\,\sin^{-1}\frac{x}{y} + \frac{t^2(x^2 - y^2)}{t\sqrt{x^2 + y^2}}$$

$$= t\left\{\sqrt{y^2 - x^2}\,\sin^{-1}\frac{x}{y} + \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}\right\} = tf(x, y)$$

$\therefore$ $f(x, y)$ is homogeneous function of degree one.

Example 6. *If* $u = \sin^{-1}\left(\dfrac{x + 2y + 3z}{\sqrt{x^8 + y^8 + z^8}}\right)$, *show that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} + 3\tan u = 0.$

Sol. Here u is not a homogeneous function.

$$\sin u = f(x, y, z) = \frac{x + 2y + 3z}{\sqrt{x^8 + y^8 + z^8}}$$

$$f(tx, ty, tz) = \frac{t\,(x + 2y + 3z)}{t^4\sqrt{x^8 + y^8 + z^8}} = t^{-3} f(x, y, z)$$

$\Rightarrow$ $\sin u$ is a homogeneous function of degree -3 in x, y, z.

$\therefore$ By Euler's theorem, we have

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) + z\frac{\partial}{\partial z}(\sin u) = -3\sin u$$

or $$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} + z\cos u\frac{\partial u}{\partial z} + 3\sin u = 0$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3\tan u = 0.$$

Example 7. (*i*) *If* $u = \cos\left(\dfrac{xy + yz + zx}{x^2 + y^2 + z^2}\right)$, *prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0.$

(*ii*) *If* $u = \log\dfrac{x^5 + y^5 + z^5}{x^2 + y^2 + z^2}$, *prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 3.$

Sol. (*i*) $$u = \cos\frac{xy + yz + zx}{x^2 + y^2 + z^2} = \cos\frac{x^2\left(\frac{y}{x} + \frac{y}{x}\cdot\frac{z}{x} + \frac{z}{x}\right)}{x^2\left[1 + \left(\frac{y}{x}\right)^2 + \left(\frac{z}{x}\right)^2\right]}$$

$$u = x^\circ \cos\frac{\frac{y}{x} + \frac{y}{x}\cdot\frac{z}{x} + \frac{z}{x}}{1 + \left(\frac{y}{x}\right)^2 + \left(\frac{z}{x}\right)^2} = x^\circ f\left(\frac{y}{x}, \frac{z}{x}\right)$$

which is a homogeneous function of x, y, z of degree 0 $\therefore$ by Euler's theorem

$\therefore$ By Euler's theorem $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0.u = 0$

Hence $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$

Aliter : Let $u = f(x, y, z)$

$\therefore$ $$f(x, y, z) = \cos \frac{xy + yz + zx}{x^2 + y^2 + z^2}$$

$$f(tx, ty, tz) = \cos \frac{t^2xy + t^2yz + t^2zx}{t^2x^2 + t^2y^2 + t^2z^2} = t^\circ \cos \frac{xy + yz + zx}{x^2 + y^2 + z^2} = t^\circ f(x, y, z)$$

$\therefore$ $f(x, y, z)$ is homogeneous function of x, y, z of degree 0.

$\therefore$ By Euler's theorem $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

(*ii*) $u = \log \dfrac{x^5 + y^5 + z^5}{x^2 + y^2 + z^2} = \log x^3 \cdot \dfrac{1 + \left(\frac{y}{x}\right)^5 + \left(\frac{z}{x}\right)^5}{1 + \left(\frac{y}{x}\right)^2 + \left(\frac{z}{x}\right)^2}$ it is not homogeneous function of x, y, z $\because$ it cannot be expressed as $x^n f\left(\frac{y}{x}, \frac{z}{x}\right)$

Now $e^u = x^3 \cdot \dfrac{1 + \left(\frac{y}{x}\right)^5 + \left(\frac{z}{x}\right)^5}{1 + \left(\frac{y}{x}\right)^2 + \left(\frac{z}{x}\right)^2}$, let $\phi(u) = e^u$ $\therefore$ $\phi(u) = x^3 f\left(\frac{y}{x}, \frac{z}{x}\right)$;

$\phi(u)$ is homogeneous function of x, y, z of degree 3 $\therefore$ by Euler's theorem

$$x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z} = 3\phi \quad \text{or} \quad x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) + z\frac{\partial}{\partial z}(e^u) = 3e^u$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3.$$

Example 8. *If* $u = \log \dfrac{x^4 + y^4}{x + y}$, *show that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3.$ (A.M.I.E. 1997)

Sol. Here u is not a homogeneous function

$$u = \log \frac{x^4 + y^4}{x + y} \Rightarrow u = \log_e\left(\frac{x^4 + y^4}{x + y}\right) \Rightarrow e^u = \frac{x^4 + y^4}{x + y}$$

which is a homogeneous function of degree 3 in x, y.

$\therefore$ By Euler's theorem, we have $x\dfrac{\partial}{\partial x}(e^u) + y\dfrac{\partial}{\partial y}(e^u) = 3 \times e^u$

or $$xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = 3e^u \quad \text{or} \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3.$$

Example 9. *If* $u = \sin^{-1} \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$, *prove that* $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\sin u \cos 2u}{4\cos^3 u}.$

Sol. $u = \sin^{-1}\dfrac{x + y}{\sqrt{x} + \sqrt{y}}$ is not a homogeneous function but

most imp 6

$\sin u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ is a homogeneous function of degree $\frac{1}{2}$ in x and y.

$\therefore$ By Euler's theorem, we have

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u$$

or $$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u \quad \text{or} \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u \qquad ...(1)$$

Differentiating (1) partially w.r.t. x,

$$x\frac{\partial^2 u}{\partial x^2} + 1.\frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial x}$$

or $$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{1}{2}\sec^2 u - 1\right)\frac{\partial u}{\partial x} \qquad ...(2)$$

Differentiating (1) partially w.r.t. y,

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} + 1.\frac{\partial u}{\partial y} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial y}$$

or $$x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = \left(\frac{1}{2}\sec^2 u - 1\right)\frac{\partial u}{\partial y} \qquad ...(3) \quad \left[\because \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}\right]$$

Multiplying (2) by x, (3) by y and adding,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \left(\frac{1}{2}\sec^2 u - 1\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= \left(\frac{1}{2\cos^2 u} - 1\right).\frac{1}{2}\tan u \qquad \text{[Using (1)]}$$

$$= -\frac{2\cos^2 u - 1}{2\cos^2 u}.\frac{1}{2}\frac{\sin u}{\cos u} = -\frac{\sin u \cos 2u}{4\cos^3 u} \qquad [\because \ 2\cos^2 u - 1 = \cos 2u]$$

Example 10. *If* $z = xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$, *show that* $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0.$

(A.M.I.E. 1997)

Sol. Let $$u = xf\left(\frac{y}{x}\right) \text{ and } v = g\left(\frac{y}{x}\right) = x^0 g\left(\frac{y}{x}\right).$$

so that $$z = u + v \qquad ...(1)$$

Since u is a homogeneous function of degree $n = 1$ in x, y

$$\therefore \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$= 0 \quad \because \quad n = 1 \qquad ...(2)$$

Since v is a homogeneous function of degree $n = 0$ in x, y

$$\therefore \quad x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2} = n(n-1)v$$

$$= 0 \quad \because \quad n = 0 \qquad ...(3)$$

Adding (2) and (3), we have

$$x^2 \frac{\partial^2}{\partial x^2}(u+v) + 2xy \frac{\partial^2}{\partial x \partial y}(u+v) + y^2 \frac{\partial^2}{\partial y^2}(u+v) = 0$$

or

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0.$$ [Using (1)]

Example 11. *Given* $z = x^n f_1\left(\frac{y}{x}\right) + y^{-n} f_2\left(\frac{x}{y}\right)$, *prove that* $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial y} + y\frac{\partial z}{\partial y} = n^2 z.$

(Marathwada 1994)

Sol. Let $u = x^n f_1\left(\frac{y}{x}\right), v = y^{-n} f_2\left(\frac{x}{y}\right)$

$\therefore$ $z = u + v$

u is a homogeneous function of x, y of degree n.

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad ...(1)$$

v is a homogeneous function of x, y of degree $-n$.

$$\therefore \quad x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = -nv \quad ...(2)$$

Diff. both (1) and (2) partially w.r.t. x and y

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = n\frac{\partial u}{\partial x} \quad ...(3)$$

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = n\frac{\partial u}{\partial y} \quad ...(4)$$

$$x\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} + y\frac{\partial^2 v}{\partial x \partial y} = -n\frac{\partial v}{\partial x} \quad ...(5)$$

$$x\frac{\partial^2 v}{\partial y \partial x} + y\frac{\partial^2 v}{\partial y^2} + \frac{\partial v}{\partial y} = -n\frac{\partial v}{\partial y} \quad ...(6)$$

Multiply (3) and (5) by x and (4), (6) by y and adding,

$$x^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}\right) + x\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) + xy\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x \partial y}\right) + xy\left(\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 v}{\partial y \partial x}\right)$$
$$+ y^2\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y^2}\right) + y\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)$$
$$= n\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) - n\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$$

$$x^2 \frac{\partial^2}{\partial x^2}(u+v) + y^2 \frac{\partial^2}{\partial y^2}(u+v) + 2xy \frac{\partial^2}{\partial x \partial y}(u+v)$$
$$+ x\frac{\partial}{\partial x}(u+v) + y\frac{\partial}{\partial y}(u+v) = n \cdot nu - n(-nv)$$

$$x^2 \frac{\partial^2 z}{\partial x^2} + y^2 \frac{\partial^2 z}{\partial y^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = n^2 z.$$

Example 12. *If* $u = \tan^{-1} \frac{y^2}{x}$, *show that* $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\sin 2u \sin^2 u.$

(P.T.U. Dec. 2002, May 2006)

Sol. $$u = \tan^{-1} \frac{y^2}{x}$$

$$\tan u = \frac{y^2}{x}$$

Let $f(x, y) = \tan u = \frac{y^2}{x} = x \frac{y^2}{x^2} = x^1 \left(\frac{y}{x}\right)^2$ which is a homogeneous function in x, y of degree 1.

$\therefore$ By Euler's theorem $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 1.f$

or $$x \frac{\partial}{\partial x}(\tan u) + y \frac{\partial}{\partial y}(\tan u) = \tan u$$

or $$x \sec^2 u \cdot \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = \tan u$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u \cos^2 u = \sin u \cos u = \frac{1}{2} \sin 2u \quad ...(1)$$

Differentiating it partially w.r.t. x and y

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = \cos 2u \cdot \frac{\partial u}{\partial x} \quad ...(2)$$

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \cos 2u \cdot \frac{\partial u}{\partial y} \quad ...(3)$$

Multiply (2) by x and (3) by y and add,

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \cos 2u \left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (\cos 2u - 1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = -2 \sin^2 u \cdot \frac{1}{2} \sin 2u$$

(using 1)

Hence $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\sin 2u \sin^2 u.$$

Example 13. *If* $u = \operatorname{cosec}^{-1} \left(\frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}\right)^{1/2}$, *prove that* $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$ $= \frac{\tan u}{144} (13 + \tan^2 u).$

(Marathwada 1990; Gujarat 1990)

Sol. $$u = \text{cosec}^{-1}\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{1/2}$$

$$f(x, y) = \text{cosec } u = \left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{1/2} = \frac{x^{1/4}\left[1+\left(\frac{y}{x}\right)^{1/2}\right]^{1/2}}{x^{1/6}\left[1+\left(\frac{y}{x}\right)^{1/3}\right]^{1/2}} = x^{\frac{1}{12}}\phi\left(\frac{y}{x}\right)$$

$\therefore$ $f(x, y)$ is a homogeneous function of degree $\frac{1}{12}$

$\therefore$ By Euler's theorem

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = \frac{1}{12}f$$

or $$x\frac{\partial}{\partial x}(\text{cosec } u)+y\frac{\partial}{\partial y}(\text{cosec } u) = \frac{1}{12}\text{ cosec } u$$

$$-x\text{ cosec } u \cot u\frac{\partial u}{\partial x} - y\text{ cosec } u\cot u\frac{\partial u}{\partial y} = \frac{1}{12}\text{ cosec } u$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{-1}{12}\text{ cosec } u \cdot \frac{1}{\text{cosec } u \cot u} = -\frac{1}{12}\tan u$$

$\therefore$ $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = -\frac{1}{12}\tan u \quad ...(1)$$

Differentiate (1) partially w.r.t. x and y

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial x\partial y} = -\frac{1}{12}\sec^2 u\frac{\partial u}{\partial x} \quad ...(2)$$

$$x\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2} = -\frac{1}{12}\sec^2 u\frac{\partial u}{\partial y} \quad ...(3)$$

Multiply (2) by x and (3) by y and add

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right) = -\frac{1}{12}\sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y} = -\left[1+\frac{1}{12}\sec^2 u\right]\left[u\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right] = -\left[\frac{12+1+\tan^2 u}{12}\right]\left[-\frac{1}{12}\right]$$

$$= \frac{13+\tan^2 u}{144}$$

Hence $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = \frac{13+\tan^2 u}{144}.$$